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COMMENT

A note on a series of Bessel functions: asymptotic and convergence properties

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Abstract. A certain series of Bessel functions—recently discussed by Lee—is an asymptotic expansion of an integral of a Bessel function. Here the asymptotic properties of the series are investigated in more detail, and it is shown that the series is not only asymptotic, but also convergent under suitable restrictions. For large positive real arguments finite numbers of terms of the series give good approximations to the integral, but the infinite sum is different from the integral.

1. Introduction

In a recent paper [1], Lee discusses some integrals of Bessel functions and gives, among others, the following expansion:

$$\int_z^\infty J_\nu(t)\,\mathrm{d}t = -\sum_{n=0}^{N} (\nu+1)(\nu+3)\ldots(\nu+2n-1)z^{-n}J_{\nu+n+1}(z) + R_N(z,\nu)$$
$$R_N(z,\nu) = (\nu+1)(\nu+3)\ldots(\nu+2N+1)\int_z^\infty t^{-N-1}J_{\nu+N+1}(t)\,\mathrm{d}t \qquad (1)$$

where the path of integration is taken parallel to the positive real axis, and ν and z are arbitrary complex numbers (if Re $\nu < -1$ the integral is not defined for real $z \leq 0$). The coefficient of the series may also be written as

$$(\nu+1)(\nu+3)\ldots(\nu+2n-1) = \frac{\Gamma[(\nu+1)/2+n]}{\Gamma[(\nu+1)/2]}2^n \qquad (2)$$

which makes apparent the fact is that is equal to unity for $n = 0$. Similar integrals were already considered by Lommel [2] and by Watson [3]; some minor errors in the formula given by Watson were pointed out by Lee [1].

It is the purpose of the present comment to investigate further the asymptotic behaviour of (1) in the limit $N \to \infty$, since: (i) the analysis given by Lee [1] is very brief and some of his relations are not quite correct; and (ii) the series has the curious property that it can be interpreted in two different (and seemingly contradictory) ways, namely as an asymptotic series for large z, and as a convergent series for any z.

2. Asymptotic properties

The characteristic property of an asymptotic series is to give an approximation to a certain function when truncated after a finite number of terms; the quality of the approximation depends on the argument and usually cannot be improved by adding more terms. This property is implicitly contained in the following formal definition of an asymptotic expansion [4, 5].

Definition. A function $f(z)$ is said to admit an asymptotic expansion in an unbounded set S of the complex plane as $z \to \infty$

$$f(z) \sim \sum_{n=0}^{\infty} a_n g_n(z) \qquad z \to \infty \quad z \in S \tag{3}$$

if the following two conditions hold: (i) the functions $g_n(z)$ form an asymptotic sequence, i.e. $g_{n+1}(z)/g_n(z) \to 0$ as $z \to \infty \, (z \in S)$, and $g_n(z) \neq 0$ for all $z \in S$ with $|z|$ sufficiently large; (ii) the partial sums $F_n(z)$ obey the condition

$$f(z) - F_n(z) = 0(g_{n+1}(z)) \qquad z \to \infty \quad z \in S \tag{4}$$

where $F_n(z) = \Sigma_{k=0}^{n} a_k g_k(z)$. The second property is equivalent to either of the following:

$$\frac{1}{g_n(z)}[f(z) - F_n(z)] \to 0 \qquad \text{as } z \to \infty \quad z \in S \tag{5a}$$

$$\frac{1}{g_n(z)}[f(z) - F_{n-1}(z)] \to a_n \qquad \text{as } z \to \infty \quad z \in S. \tag{5b}$$

It is quite easy to verify that the series in question, equation (1), satisfies both conditions (i) and (ii). The functions $g_n(z) = z^{-n} J_{\nu+n+1}(z)$ form an asymptotic sequence in any set $S_\alpha = \{z | \alpha \leqslant |\arg z| \leqslant \pi - \alpha\}$ for arbitrary positive $\alpha < \pi/2$, since all the zeros of $J_\nu(z)$ are either real or lie in a bounded strip parallel to the real axis [3], and since the ratio $J_{\nu+1}(z)/J_\nu(z)$ is bounded as $z \to \infty \, (z \in S_\alpha)$. Either of (4), (5$a$), (5$b$) may be verified by using the leading term of the asymptotic expansion of Bessel functions of large arguments

$$J_\nu(z) \sim (2/\pi z)^{1/2} \cos[z - (\pi/4)(2\nu + 1)] \qquad z \to \infty \quad |\arg z| < \pi \tag{6}$$

and by integration by parts.

Although the asymptotic expansion of a function in a certain domain—if it exists—is unique, an asymptotic expansion does not uniquely define a function: there may be non-zero functions whose asymptotic expansion is identically zero, and two functions whose difference *is* such a function obviously have the same asymptotic expansion. In our particular case, the asymptotic expansion of any function $p(z)$ having at most a pole at infinity (i.e. $p(z) = \mathrm{O}(z^k)$, $z \to \infty$, $k \in \mathbb{N}$) vanishes identically. Therefore one may write

$$-\sum_{n=0}^{\infty} (\nu+1)(\nu+3)\ldots(\nu+2n-1) z^{-n} J_{\nu+n+1}(z) \sim p(z) + \int_z^{\infty} J_\nu(t)\, \mathrm{d}t. \tag{7}$$

Since both the partial sums and the integral in (7) increase exponentially with increasing z off the real axis, the function $p(z)$ becomes arbitrarily small compared with both of them for sufficiently large z.

For positive real z, the series (7) is not an asymptotic expansion in the sense of the definition above. The functions $g_n(z)$ have zeros and therefore do not form an asymptotic sequence; (4) does not hold, and the limits in (5) do not exist. Nevertheless, the remainder $R_N(z, \nu)$ does obey the condition

$$R_N(z, \nu) = \mathrm{O}(z^{-N-3/2}) \qquad z \to +\infty \tag{8}$$

which can be shown similarly by integration by parts. This condition is even stronger than the one required for asymptotic power series, and for large real z the series truncated after a finite number of terms gives as good an approximation to the integral as any asymptotic power series. We are therefore justified in calling this series 'asymptotic' in a more general sense than the definition above, and to denote it by a suitable symbol, say $\dot{\sim}$, which we define as follows

$$f(z) \dot{\sim} \sum_{n=0}^{\infty} a_n g_n(z) \quad \text{as} \quad z \to \infty \qquad \text{if } f(z) - F_n(z) = O(z^{-N-1}). \tag{9}$$

If $g_n(z) = z^{-n}$, this definition is identical to the standard definition of an asymptotic expansion.

3. Convergence properties

If the series expansion of a certain function

$$f(z) = \sum_{n=0}^{\infty} a_n g_n(z) \tag{10}$$

converges in an unbounded set S for sufficiently large z, then this series is automatically the asymptotic expansion of $f(z)$ if the functions $g_n(z)$ form an asymptotic sequence. However, the generalised asymptotic property (9) does *not* follow from (10). In the following we establish the conditions of convergence of the series in question, and we shall see that (9) and (10) hold for the same series but for different functions: the series is a (generalised) asymptotic expansion of one function, and the sum of the series is a different function.

The asymptotic properties of Bessel functions of large order can be inferred from the first term of their power series expansions [3, 6]

$$J_\nu(z) \sim (z/2)^\nu \frac{1}{\Gamma(\nu+1)} \qquad \nu \to \infty. \tag{11}$$

From (1), (2) and (11), one gets the following expression for the terms of the series in the limit $n \to \infty$:

$$(\nu+1)(\nu+3)\ldots(\nu+2n-1)z^{-n}J_{\nu+n+1}(z) \sim \frac{z^{\nu+1}}{\Gamma(\nu+1)/2]} \frac{1}{\nu+n+1} \frac{\Gamma[n+(\nu+1)/2]}{\Gamma(n+\nu+1)}. \tag{12}$$

It is a sufficient condition of convergence that the terms decrease faster than $1/n$. This is evidently fulfilled if $\operatorname{Re}[(\nu+1)/2] < \operatorname{Re}(\nu+1)$, i.e. for $\operatorname{Re}\nu > -1$ the series actually converges†, and the limit $R_N(z, \nu)$ for $N \to \infty$ exists and is finite. We now proceed to evaluate this limit. First we note that for $\operatorname{Re}\nu > -1$

$$\int_z^{\infty} t^{-N-t} J_{\nu+N+1}(t)\,\mathrm{d}t \sim \int_0^{\infty} t^{-N-1} J_{\nu+N+1}(t)\,\mathrm{d}t \qquad N \to \infty. \tag{13}$$

This can be shown by taking $\int_z^\infty = \int_0^\infty - \int_0^z$. The first inegral is given by Watson [3]

$$\int_0^{\infty} t^{\mu-\nu-1} J_\nu(t)\,\mathrm{d}t = \frac{\Gamma(\mu/2)}{\Gamma(\nu-\mu/2+1)} 2^{\mu-\nu-1} \qquad 0 < \operatorname{Re}\mu < \operatorname{Re}\nu + \tfrac{1}{2}. \tag{14}$$

† More precisely, considered as a function of z, the series converges uniformly in any domain excluding the point at infinity (Weierstrass M-test for uniform convergence). It is also evident that the series diverges for $\operatorname{Re}\nu < -1$ (unless ν is an odd integer). We can say nothing about the convergence for $\operatorname{Re}\nu = -1$, except the trivial case $\nu = -1$, where all terms (but one) of the series vanish identically.

By inserting the asymptotic formula (11) into the integral from 0 to z one can show that for sufficiently large N and for Re $\nu > -1$ this integral becomes arbitrarily small compared with the integral from 0 to ∞. From (1), (2), (13) and (14) we get

$$R_N(z,\nu) \sim \frac{\Gamma(N+\nu+3/2)}{\Gamma[(\nu+1)/2]} 2^{N+1} \int_0^\infty t^{-N-1} J_{\nu+N+1}(t)\,\mathrm{d}t = 1 \qquad N \to \infty. \tag{15}$$

In the particular case $\nu = 1$ this result can also be obtained by computing the integral from z to ∞ analytically. In the limit $N \to \infty$ the remainder is independent of ν and z and equal to one.

In summary, we have obtained the following results:

$$-\sum_{n=0}^{\infty} (\nu+1)(\nu+3)\ldots(\nu+2n-1) z^{-n} J_{\nu+n+1}(z)$$
$$\sim p(z) + \int_z^\infty J_\nu(t)\,\mathrm{d}t \qquad z \to \infty \quad z \in S_\alpha = \{z \mid \alpha \leqslant |\arg z| \leqslant \pi - \alpha\}$$
$$0 < \alpha < \pi/2 \quad \nu \in \mathbb{C} \quad p(z) = \mathrm{O}(z^k) \quad k \in \mathbb{N} \tag{16a}$$

Table 1. Partial sums of the series (1) for $\nu = 1$ (denoted by $F_N(z)$) for various values of z and N compared to $J_0(z)$.

z	$J_0(z)$	N	$F_N(z)$	$J_0(z) - F_N(z)$
0.5	0.938 469 807	1	−0.040 858 943	0.979 328 751
		2	−0.046 002 511	0.984 472 318
		100	−0.060 917 634	0.999 387 441
		1 000	−0.061 467 819	0.999 937 627
		10 000	−0.061 523 944	0.999 993 751
5	−0.177 596 771	1	−0.192 497 609	0.014 900 837
		2	−0.317 691 964	0.140 095 193
		10	−0.765 130 716	0.587 533 945
		100	−1.118 144 625	0.940 547 854
		1 000	−1.171 378 640	0.993 781 869
		10 000	−1.176 972 091	0.999 375 320
		100 000	−1.177 534 275	0.999 937 503
24	−0.056 230 274	5	−0.056 214 713	−0.000 015 562
		10	−0.056 224 723	−0.000 005 551
		15	−0.056 211 453	−0.000 018 821
		20	−0.056 481 704	0.000 251 430
		100	−0.297 553 146	0.241 322 872
		1 000	−0.922 358 017	0.866 127 743
		10 000	−1.041 936 286	0.985 706 012
		100 000	−1.054 791 339	0.998 561 065
100	0.019 985 850	10	0.019 985 850	0.000 000 000
		100	0.019 985 850	0.000 000 000
		250	0.019 945 828	0.000 040 022
		1 000	−0.062 253 322	0.082 239 173
		10 000	−0.758 851 434	0.778 837 284
		100 000	−0.955 324 546	0.975 310 397
		1000 000	−0.977 517 277	0.997 503 127

$$\dot{\sim} \int_z^\infty J_\nu(t)\,\mathrm{d}t \qquad z \to +\infty \quad z \in \mathbb{R} \quad \nu \in \mathbb{C} \tag{16b}$$

$$= -1 + \int_z^\infty J_\nu(t)\,\mathrm{d}t \qquad z \in \mathbb{C} \quad \mathrm{Re}\,\nu > -1. \tag{16c}$$

Note that (16*b*) and (16*c*) do *not* contradict each other! (Equation (16*c*) does contradict Lee's (13), which is incorrect.) For any positive real z, up to a certain number of terms (as long as $J_{\nu+n+1}(z)$ is an oscillating function of n) the partial sums approach the value of the integral (16*b*) (and come very close to it for large z); when more terms are added the Bessel function becomes a monotonic function of n, and the series eventually converges very slowly towards a different value, namely the integral minus one. For $\nu = 1$ (where the integral is equal to $J_0(z)$) this curious behaviour is illustrated by some numerical examples in table 1.

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